Counting

Product Rule

Suppose we are trying to determine the number of possible pairs of tasks, given that there are n_1 outcomes of the first task, and also after the first task has been complete, there are n_2 possible outcomes of the second task. Then there are $n_1 n_2$ possible outcomes for the pair of tasks.

Example: How many different two[l.c.] letter passcodes are there if both letters have to be different?

Solution: Think of as two tasks

Choose 1st letter	Choose 2nd letter
$n_1 = 26$	$n_2 = 25$

Total # of passcodes $= n_1 n_2 = (26)(25)$.

General Product Rule

Proposition. Suppose $k^{\geq 1}$ tasks are to be done, and there are n_1 outcomes for the first task, and there are n_j outcomes for the j^{th} task, given that the first $(j-1)$ tasks have been completed, for $2 \leq j \leq k$. Then there are $n_1 n_2 \dots n_k$ outcomes for the sequence of tasks.

Proof: We will prove this by induction on k.

Base case: for $k=1$, there are n_1 possible outcomes, so the statement is true.

(Induction hypothesis) Induction step: Assume that the statement holds for j tasks, with $j \geq 1$.

We consider the case of $j+1$ tasks..

Let the first task T be the first j tasks, and let Q be the $(j+1)^{st}$ task. By the Product Rule, The # of outcomes is

$$n_T n_Q$$

, where n_T = # of outcomes for the first j tasks, and $n_Q = n_{j+1}$ = # of outcomes for the $(j+1)^{st}$ task.

By the induction hypothesis,

$$n_T = n_1 n_2 \cdots n_j .$$

$\therefore$ # of outcomes for $j+1$ tasks

is $n_T n_Q = n_1 n_2 \cdots n_j n_{j+1}$.

By induction, the statement is true for any $k \in \mathbb{N}$. ∎

Example A binary string of length p is a sequence of p digits, where each digit is 0 or 1.

eg. 010111 is a binary string of length 6.

Question: How many binary strings of length 8 are there that are not all 0's or all 1's?

Solution: Think of digit as a task. By the General Product Rule,

The answer is $\underset{2}{n_1}\,\underset{2}{n_2}\cdots\underset{2}{n_8} - 2$ ← not counting 11111111 or 00000000

Ans $2^8 - 2 = \boxed{254}$.

Question: How many different ways can we select 3 officers (CEO, CFO, CTO) from our class of 33 students?

frog

Ans: $33 \cdot 32 \cdot 31 = 32{,}736$

(# permutations of n objects)
= (# of arrangements of n objects in a row)
= (# of 1-1, onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$)

$$\underset{1}{\underline{n}} \quad \underset{2}{\underline{(n-1)}} \quad \underset{3}{\underline{(n-2)}} \quad \underline{\quad} \cdots \quad \underset{n}{\underline{1}}$$

slots

$\therefore \; = \boxed{n!}$.

Symbol for this ${}_nP_n =$

n objects → With r slots:

$$\underset{1}{\underline{n}} \quad \underset{2}{\underline{(n-1)}} \cdots \underset{r}{\underline{\overset{n-r+1}{n-(r-1)}}}$$

$$\boxed{n(n-1)\cdots(n-r+1) = {}_nP_r}.$$

Sum Rule Suppose there are n_1 outcomes for task T_1 & there are n_2 outcomes for Task T_2, and these tasks are independent. Then the number of outcomes for (T_1 or T_2) is $n_1 + n_2$.

Ex How many binary numbers are between 5 and 7 digits long?

General Sum Rule: If tasks $T_1, \ldots, T_k$ have $n_1, n_2, \ldots, n_k$, respectively, possible outcomes, then (T_1 or T_2 or T_3 or ... or T_k) has $n_1 + n_2 + \ldots + n_k$ possible outcomes (can prove by induction)

must start with 1

T_1 = # of 5-digit binary #s: $\underline{1} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2}$

T_2 = # of 6-digit binary #s $\underline{1} \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

T_3 = # of 7-digit binary #s $1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

By Generalized Sum Rule, answer: $2^4 + 2^5 + 2^6$

$= 16 + 32 + 64 = \boxed{112}$.

$\frac{00099}{100}$